

**Comment on “Modified F(R) Hořava-Lifshitz gravity: a way to  
accelerating FRW cosmology” by M. Chaichian, S. Nojiri, S. D.**

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**Abstract**

The partial Hamiltonian analysis of the Hořava-type action presented in the paper by M. Chaichian, S. Nojiri, S. D. Odintsov, M. Oksanen, A. Tureanu (*Class. Quant. Grav.* **27** (2011) 185021) is incorrect; for the authors’ choice of variables, a covariant shift, instead of a contravariant shift which is the one usually used in General Relativity (GR) in ADM variables, the true algebra of constraints differs from what they presented. The algebra of constraints for their choice of variables is explicitly given for GR and compared with the standard algebra.

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## I. INTRODUCTION

Hořava-type models have become a very popular subject and it is the custom, as in the original paper [1], to present some elements of the Hamiltonian analysis; in this respect paper [2] is no exception (see section 3). In Hořava-type models, the full diffeomorphism of the Einstein theory (and Einstein's main idea: general covariance) is abandoned. This deformation of General Relativity is a topic of continuing discussion, with disparate opinions.

The Hamiltonian formulation of singular systems can be obtained through a general procedure that can be applied to covariant and non-covariant systems. When considering Hořava-type models, the residual GR properties and the rules of Dirac's procedure should not be abandoned, e.g. the dependence of Dirac's procedure on the choice of independent variables, and the fact that raising and lowering indices in GR are performed by fields (of course, for Hořava-type models only spatial indices are to be raised and lowered).

In paper [2] the Hamiltonian formulation of the Hořava-type model is performed using ADM variables, but in a form that is non-standard for GR ( $N_k$  is taken as an independent variable, in contrast to the standard choice of  $N^k$ ). This departure from convention must affect the Hamiltonian formulation and the corresponding algebra of constraints. Calculations of some Poisson Brackets are provided to illustrate that the result does not simply mimic the standard one. The complete algebra of secondary first-class constraints of GR in ADM variables for the standard choice ( $N^k$ ) is also compared with the one calculated for the choice of variables used in [2].

## II. RE-EXAMINATION OF HAMILTONIAN ANALYSIS OF THE MODEL OF [2]

The Hamiltonian analysis (partial) of the modified Hořava-type model (see (C27))<sup>1</sup> was performed in [2]. After the elimination of the pair of second-class constraints and the corre-

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<sup>1</sup> We use (C##) to refer to equations of [2].

sponding pair of phase-space variables  $(A, \pi_A)$ , the total Hamiltonian (C35)<sup>2</sup> was obtained:

$$H_T = \dot{N}\pi + \int d\mathbf{x} \dot{N}_i \pi^i + N \int d\mathbf{x} H_0 + \int d\mathbf{x} N_i H^i. \quad (1)$$

Note that the independent variables (and conjugate momenta) of this formulation are  $N(\pi)$ ,  $N_i(\pi^i)$ ,  $g_{pq}(\pi^{pq})$ ,  $A(\pi_A)$  and  $B(\pi_B)$ . For this non-standard choice of variables (compared with GR in ADM variables) the fundamental Poisson Brackets (PBs) are (see (C31) for the full set)

$$\{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}), \quad \{N_i(\mathbf{x}), \pi^j(\mathbf{y})\} = \delta_i^j \delta(\mathbf{x} - \mathbf{y}). \quad (2)$$

In the projectable case,  $N = N(t)$ , the closure of the constraint algebra is assumed to be of the form (see (C43))

$$\{\Phi_0, \Phi_0\} = 0, \quad \{\Phi_S(\xi_i), \Phi_0\} = 0, \quad \{\Phi_S(\xi_i), \Phi_S(\eta_j)\} = \Phi_S(\xi^j \partial_j \eta_i - \eta^j \partial_j \xi_i) \approx 0, \quad (3)$$

where

$$\Phi_0 = \int d\mathbf{x} H_0, \quad \Phi_S(\xi_i) = \int d\mathbf{x} \xi_i H^i. \quad (4)$$

Equations (3) are similar to those presented by Hořava [1], which are based on the results known for GR in ADM variables with the additional projectability condition,  $N = N(t)$ ; but in the original Hořava paper [1] the constraint algebra was written for the standard form of shift variable,  $N^k$ , and its conjugate momentum,  $\pi_k$  (the primary constraint), that leads to the secondary constraint  $H_k$ , which is related to  $H^i$  of [2] by

$$H_k = g_{ik} H^i. \quad (5)$$

Relation (5) differs from that for an ordinary field theory where the indices are raised and lowered by the Minkowski tensor without affecting the calculation of the PBs. Further,  $\xi_i$

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<sup>2</sup> Instead of using undetermined Lagrange multipliers in front of the primary constraints in (1), we write the undetermined velocities as they appear in the Legendre transformation. In the case under consideration this difference is not important, but for some formulations it is crucial for the restoration of the gauge transformations of all fields of an action.

and  $\eta_i$  in (3)-(4) are test functions that appear in smeared or global constraints. Based on the distributional properties of delta functions, they merely allow a different formal presentation of the PBs of two constraints (e.g. see section 3 of [3]). The test functions are assumed to have a zero PB with all of the variables present in the constraints, but this property would be lost if one were to use  $g_{ik}\xi^k$  instead of  $\xi_i$ . To state this fact more explicitly, when shift is treated as a test function in the calculations, the result depends on which variables are defined as independent (with the fundamental PBs (2)). For example, for the fundamental PB of (2):  $\{N_i(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = 0$ . But for the dependent variable  $N^p$ , one obtains

$$\{N^p(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = N_q \{g^{pq}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = -\frac{1}{2} (N^l g^{pk} + N^k g^{pl}) \delta(\mathbf{x} - \mathbf{y}) \neq 0. \quad (6)$$

Even without a change of independent variables, if some constraints (e.g.  $H_k$ ) are linear combinations of others (e.g.  $g_{ik}H^i$ ), with field-dependent coefficients (fields that are canonically conjugate to variables present in constraints), then the constraint algebra cannot be the same (for a more detailed example see section 4 of [3]). Such a change does not affect the closure of the algebra, but the form of closure must be different. Based on the general arguments for the choice of parametrisation in [2], which have just been described, the algebra must differ from what is asserted.

To illustrate this general point we perform simple calculations for the model of [2]. The total Hamiltonian (1) has the following so-called Hamiltonian and momentum constraints (see (C34)):

$$H_0 = \frac{1}{\sqrt{g}} \left[ \frac{1}{B} \left( g_{ik} g_{jl} \pi^{ij} \pi^{kl} - \frac{1}{3} g_{ij} g_{kl} \pi^{ij} \pi^{kl} \right) - \frac{1}{3\mu} g_{pq} \pi^{pq} \pi_B - \frac{1-3\lambda}{12\mu^2} B (\pi_B)^2 \right] \quad (7)$$

$$+ \sqrt{g} [B (E^{ij} G_{ijkl} E^{kl} + A) - F(A) + 2\mu g^{ij} \nabla_i \nabla_j B],$$

$$H^i = -2\partial_j \pi^{ij} - g^{ij} (2\partial_k g_{jl} - \partial_j g_{kl}) \pi^{kl} + g^{ij} \partial_j B \pi_B, \quad (8)$$

where the elimination of the second-class constraints<sup>3</sup> gives  $A = \tilde{A}(B)$ . Consider the PB,  $\{H^i(\mathbf{x}), \int d\mathbf{y} H_0(\mathbf{y})\}$ , that is claimed to be zero in [2] (where the projectable case is discussed); the PB for one particular term of  $H_0$  (second in (7)), i.e.

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<sup>3</sup> To simplify the notation in this comment we omit the superscript <sup>(3)</sup>, i.e.  $g_{km}^{(3)} = g_{km}$ ,  $\sqrt{g^{(3)}} = \sqrt{g}$ .

$$\left\{ H^i(\mathbf{x}), \int d\mathbf{y} \left( -\frac{1}{3\mu} \right) \left( \frac{1}{\sqrt{g}} g_{pq} \pi^{pq} \pi_B \right) (\mathbf{y}) \right\}, \quad (9)$$

produces non-zero contributions, which can be immediately seen for the PB of the last term of the momentum constraint (8):

$$-\frac{1}{3\mu} (\partial_j B \pi_B)(\mathbf{x}) \int d\mathbf{y} \left[ \{g^{ij}(\mathbf{x}), \pi^{pq}(\mathbf{y})\} \left( \pi_B \frac{1}{\sqrt{g}} g_{pq} \right) (\mathbf{y}) \right] = \frac{1}{3\mu} \frac{1}{\sqrt{g}} g^{ij} \partial_j B (\pi_B)^2(\mathbf{x}). \quad (10)$$

Note: all other contributions of (9) are linear in momentum  $\pi_B$  and cannot compensate (10), which is proportional to  $(\pi_B)^2$ . There is only one term in the Hamiltonian constraint (third term of (7)) that might contribute a  $(\pi_B)^2$  term, but it enters (7) with a different numerical coefficient. If contribution (10) could be compensated, it would place a restriction on the coefficients (i.e. a particular relation between  $\mu$  and  $\lambda$ ). Yet, the PB of the momentum constraint with the third term of (7) is

$$\left\{ H^i(\mathbf{x}), \int d\mathbf{y} \left( -\frac{1-3\lambda}{12\mu^2} \right) \left( \frac{1}{\sqrt{g}} B \pi_B^2 \right) (\mathbf{y}) \right\} = 0, \quad (11)$$

and compensation of (10) is impossible, even with a restriction on the parameters  $\mu$  and  $\lambda$ ; therefore, the claim made by the authors that PB (3) is zero for their choice of variables ( $N_i$  and the corresponding momentum constraint  $H^i$  (8)) is incorrect.

The structure of contribution (10) and the zero value of PB (11) preclude the result that (9) is proportional to the Hamiltonian constraint. To have closure on the secondary constraint, only a proportionality to the momentum constraint can be expected. Indeed, the complete calculation of (9) confirms such an expectation, i.e.

$$\left\{ H^i(\mathbf{x}), \int d\mathbf{y} \left( -\frac{1}{3\mu} \right) \left( \frac{1}{\sqrt{g}} g_{pq} \pi^{pq} \pi_B \right) (\mathbf{y}) \right\} = \frac{1}{3\mu} \frac{\pi_B}{\sqrt{g}} H^i(\mathbf{x}). \quad (12)$$

Instead of calculating a complete algebra of constraints for model [2], we show how the authors' choice of variables affects the algebra of constraints for the Hamiltonian of GR in the ADM variables (to the best of our knowledge, such an algebra was not reported). For the standard choice of ADM variables, the total Hamiltonian is

$$H_T = \int d\mathbf{x} \left( \dot{N} \pi + \dot{N}^i \pi_i + N H_0 + N^i H_i \right),$$

and its algebra is well-known (e.g. see Castellani's paper [4]), i.e.

$$\left\{ H_0, \int d\mathbf{y} N H_0 \right\} = N_{,r} g^{rm} H_m + (N g^{rm} H_m)_{,r} , \quad (13)$$

$$\left\{ H_0, \int d\mathbf{y} N^k H_k \right\} = (N^k H_0)_{,k} , \quad (14)$$

$$\left\{ H_k, \int d\mathbf{y} N H_0 \right\} = N_{,k} H_0 , \quad (15)$$

$$\left\{ H_k, \int d\mathbf{y} N^m H_m \right\} = N_{,k}^m H_m + (N^m H_k)_{,m} . \quad (16)$$

We choose to work with PBs of this form, (13)-(16) (without test functions), because it is the form used in the demonstration of the closure of Dirac's procedure on the secondary constraints; further, this form is needed for the construction of the generator to restore the gauge transformations of *all* fields, irrespective of how they were named (dynamical, non-dynamical, multipliers, etc.). The gauge transformations that follow from the Hamiltonian formulation should be the same as the Lagrangian transformations found for *all* fields.

The total Hamiltonian for the choice of shift in [2] becomes

$$H_T = \int d\mathbf{x} \left( \dot{N} \pi + \dot{N}_i \pi^i + N H_0 + N_i H^i \right) , \quad (17)$$

and although the constraint algebra is also closed, it is more complicated than it would be for the standard choice:

$$\left\{ H_0, \int d\mathbf{y} N H_0 \right\} = N_{,k} H^k + (N H^k)_{,k} , \quad (18)$$

$$\left\{ H_0, \int d\mathbf{y} N_k H^k \right\} = (N_k g^{ki} H_0)_{,i} + 2 \frac{1}{\sqrt{g}} G_{pqab} \pi^{pq} g^{ka} N_k H^b , \quad (19)$$

$$\left\{ H^k, \int d\mathbf{y} N H_0 \right\} = g^{ki} N_{,i} H_0 - 2 N \frac{1}{\sqrt{g}} g^{kp} G_{abpq} \pi^{ab} H^q , \quad (20)$$

$$\left\{ H^k, \int d\mathbf{y} N_m H^m \right\} = -\partial_i (N_m g^{km} H^i) - \partial_i N_m g^{km} H^i + N_m g^{kn} g^{mj} (\partial_n g_{ji} - \partial_j g_{ni}) H^i . \quad (21)$$

Note: if for the standard choice of variables only (13) has a field-dependent structure "constant", then for the choice in [2] the situation is the opposite; all PBs (19)-(21), with

the exception of (18), have field-dependent structure “constants”, and they are very complicated. Imposing the projectability condition leads to the elimination of terms with spatial derivatives of lapse, and of all total spatial derivatives in (13)-(14) and (18)-(19). For the standard choice of variables one obtains the Hořava algebra [1], but for the non-standard choice of [2], the algebra is different. One can observe that the form of contribution (12) is consistent with the second term of (20).

### III. CONCLUSION

The Hamiltonian formulation of singular systems is sensitive to the choice of field parametrisation [5] that, in particular, affects the PBs of constraints with the total Hamiltonian and among themselves. Moreover, according to the Dirac conjecture, *all* first-class constraints generate gauge transformations [6] and a knowledge of their PB algebra is important for the construction of the gauge generator (e.g. see [4]).

In [2] (also see papers by the same authors [7–9], and by others, e.g. [10]) the non-standard choice of shift function was made: instead of  $N^k$ ,  $N_k$  was chosen as the independent variable. Unlike ordinary field theories, even the choice of covariant or contravariant fields (because another *field*, the metric  $g_{km}$ , is used to raise or lower indices) drastically affects the results of PBs of constraints with the total Hamiltonian and among themselves. The assumption made in [2] that the algebra of constraints is the same for both choices of shift function, is incorrect. For example, (10) is obviously non-zero, contrary to the claim of [2]. In the case of GR written in ADM variables, the well known algebra of constraints (13)-(16), which is always written for the standard choice of shift ( $N^k$ ) function, becomes much more complicated for the choice of  $N_k$  in [2], see (18)-(21).

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